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A Result in Best Approximation Theory

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Using a fixed-point theorem of G. Jungck [*Math. Mag.* **49**, No. 1 (1976), 32–34], we generalize a result of S. P. Singh [*J. Approx. Theory* **25** (1979), 84–90] on best approximation.

Let X be a normed linear space. A mapping $T: X \to X$ is contractive on X (resp. on a subset C of X) if $\|Tx - Ty\| \leqslant \|x - y\|$ for all x, y in X (resp. C). The set of fixed points of T in X is denoted by $F(T)$. If $\bar{x}$ is a point of X, the set D of best C-approximants to $\bar{x}$ consists of the points y in C such that $\|y - \bar{x}\| = \inf\{\|z - \bar{x}\| : z \in C\}$. A subset C of X is said to be starshaped with respect to a point $q \in C$ if, for all x in C and all λ, $0 \leqslant \lambda \leqslant 1$, $\lambda x + (1 - \lambda) q$ is in C. A convex set is starshaped with respect to each of its points.

Singh [5], relaxing the linearity of the operator T and the convexity of D in the original statement of the well-known result of Brosowski [1], proved the following

THEOREM 1. *Let $T: X \to X$ be a contractive operator on X. Let C be a T-invariant subset of X and let $\bar{x} \in F(T)$. If $D \subseteq X$ is nonempty, compact, and starshaped, then $D \cap F(T) \neq \varnothing$.*

Singh [6] observed that only the nonexpansiveness of T on $D' = D \cup \{\bar{x}\}$ is necessary. Further, Hicks and Humphries [2] stressed that

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a point $y \in D$ is not necessarily in the interior of C, i.e., $y \in \partial C$. Then the assumption $T: C \to C$ can be weakened to the condition $T: \partial C \to C$.

In the current terminology of fixed-point theory, a contractive operator is called nonexpansive. Park [4], relativizing the concept of nonexpansiveness of T with respect to another mapping $I: X \to X$, introduced the inequality

$$\|Tx - Ty\| \leqslant \|Ix - Iy\| \tag{1}$$

for all x, y in X. Of course, T is continuous whenever I is continuous.

Jungck [3] proved that

THEOREM 2. *Let (X, d) be a compact metric space and $T, I: X \to X$ be two commuting mappings such that $T(X) \subseteq I(X)$, I is continuous, and $d(Tx, Ty) < d(Ix, Iy)$* whenever *$Ix \neq Iy$. Then $F(T) \cap F(I)$ is singleton.*

By using this theorem, we generalize Theorem 1 with the following result.

THEOREM 3. *Let $T, I: X \to X$ be operators, C be a subset of X such that $T: \partial C \to C$, and $\bar{x} \in F(T) \cap F(I)$. Further, T and I satisfy (1) for all x, y in D' and let I be linear, continuous on D, and $ITx = TIx$ for all x in D. If D is nonempty, compact and starshaped with respect to a point $q \in F(I)$ and if $I(D) = D$, then $D \cap F(T) \cap F(I) \neq \varnothing$.*

Proof. Let $y \in D$ and hence Iy is in D since $I(D) = D$. Further, $y \in \partial C$ and then Ty is in C since $T(\partial C) \subseteq C$. From (1), it follows that

$$\|Ty - \bar{x}\| = \|Ty - T\bar{x}\| \leqslant \|Iy - I\bar{x}\| = \|Iy - \bar{x}\|$$

and therefore Ty is in D. Thus T maps D into itself.

Let $\{k_n\}$ be a sequence of real numbers such that $0 \leqslant k_n < 1$ and converging to 1. Define a sequence $\{T_n\}$ of mappings by putting

$$T_n x = k_n \cdot Tx + (1 - k_n) \cdot q$$

for all x in D and for each n. Since D is starshaped with respect to q, we have that T_n maps D into itself for each n. Since I is linear and commutes with T on D, we have

$$\begin{aligned} T_n Ix &= k_n \cdot TIx + (1 - k_n) \cdot Iq = k_n \cdot ITx + (1 - k_n) \cdot Iq \\ &= I(k_n \cdot Tx + (1 - k_n) \cdot q) = IT_n x \end{aligned}$$

for all x in D. Thus I commutes with T_n on D for each n and $T_n(D) \subseteq D = I(D)$. Further, we have that

$$\|T_n x - T_n y\| = k_n \cdot \|Tx - Ty\| \leqslant k_n \cdot \|Ix - Iy\| < \|Ix - Iy\|$$

whenever $Ix \neq Iy$. Since D is compact and I is continuous, we deduce that $F(T_n) \cap F(I) = \{x_n\}$ for each n by Theorem 2. Once again the compactness of D ensures that $\{x_n\}$ has a convergent subsequence $\{x_{n(i)}\}$ to a point z in D. Since

$$x_{n(i)} = T_{n(i)} x_{n(i)} = k_{n(i)} \cdot T x_{n(i)} + (1 - k_{n(i)}) \cdot q$$

and T is continuous, we have, as $i \to \infty$, that $z = Tz$, i.e., $z \in D \cap F(T)$.

Further, the continuity of I implies that

$$Iz = I(\lim_{i \to \infty} x_{n(i)}) = \lim_{i \to \infty} I x_{n(i)} = \lim_{i \to \infty} x_{n(i)} = z,$$

i.e., $z \in F(I)$ and therefore the thesis.

Of course, Theorem 1 is a consequence of Theorem 3 assuming $I =$ identity on X.

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